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LETTER TO THE EDITOR

The correct form of the single-eigenvalue distribution for the non-zero mean-matrix ensembles

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Abstract. The exact N-dimensional joint eigenvalue distribution is used to show that the single-eigenvalue distribution $P(E)$ is of the form $f(E-M_0/N)$ where M_0/N is the mean value of matrix elements. It is further shown that the distribution can be written as Wigner's distribution centred at M_0/N plus correction terms of the order of N^{-1}.

Recently there has been considerable interest in the distribution of single eigenvalues for matrix ensembles in which the mean value of the matrix elements is non-zero. Two very different kinds of distribution have been reported in the literature, one obtained by Edwards and Jones (1976) using what is known as $n\to 0$ technique and the other by Kota and Potbhare (1977) using the method of moments. Later it was pointed out by Jones *et al* (1978) that the distribution given by Kota and Potbhare is in error because certain correlations have not been correctly taken into account. In a private communication Parikh has pointed out that these correlations are not so strong as to change the distribution of single eigenvalues drastically. The purpose of the present note is to look at this problem from a different point of view, and determine what is the correct form of this distribution. This different viewpoint is provided by first deriving the joint eigenvalue distribution for the matrix ensembles having a non-zero mean.

Following Edwards and Jones (1976) let us consider a matrix ensemble, each element of which has the distribution

$$p(H_{ij})=\left(\frac{2\pi J^2}{N}\right)^{-1/2}\exp\left[-\frac{N}{2J^2}\left(H_{ij}-\frac{M_0}{N}\right)^2\right]. \tag{1}$$

The joint eigenvalue distribution $P(\{E_\mu\})$ is then given by (Ullah 1979)

$$P(\{E_\mu\})=K\exp\left(-\frac{N}{4J^2}\sum E_\mu^2+\frac{M_0}{J^2}\sum E_\mu\right)$$
$$\times\prod_{\mu<\nu}|E_\mu-E_\nu|\int \mathrm{d}\bar{T}\exp\left[\frac{M_0}{J^2}\sum_k E_k\left(\sum_{\mu<\nu}T_{k\mu}T_{k\nu}\right)-\frac{N}{4J^2}\sum_\mu\left(\sum_k E_kT_{k\mu}^2\right)^2\right] \tag{2}$$

where K is the normalisation constant, $\mathrm{d}\bar{T}$ is the invariant volume element in the space of orthogonal matrices and $T_{k\mu}$ are the elements of the orthogonal matrix. Using orthonormality relations on T_k, expression (2) can be rewritten as

$$P(\{E_\mu\}) = K \exp\left[-\frac{N}{4J^2}\sum_\mu \left(E_\mu - \frac{M_0}{N}\right)^2\right]$$
$$\times \prod_{\mu<\nu} |E_\mu - E_\nu| \int d\bar{T} \exp\left\{\frac{M_0}{J^2}\sum_k \left(E_k - \frac{M_0}{N}\right)\left(\sum_{\mu<\nu} T_{k\mu}T_{k\nu}\right)\right.$$
$$\left. -\frac{N}{4J^2}\sum_\mu \left[\sum_k \left(E_k - \frac{M_0}{N}\right) T_{k\mu}^2\right]^2\right\}. \tag{3}$$

Introducing the new variables $\epsilon_\mu = E_\mu - M_0/N$, we find that if we formally integrate over $(N-1)$ of ϵ_μ and $d\bar{T}$, then the probability density of the single eigenvalue $P(E)$ will be a function of the form $f(E - M_0/N)$. One can further check that this is indeed the form by carrying out explicit integrations in $N=2$ and 3 dimensions (Ullah 1979). Since neither the distribution given by Edwards and Jones (1976) nor that given by Kota and Potbhare (1977) are of this form, they cannot be the correct distributions.

We next wish to determine the function f. For this purpose we separate the exponent under the integral sign in expression (3) in the following way using probability theory (Kendall 1945):

$$T_{k\mu}^2 = \langle T_{k\mu}^2\rangle + (T_{k\mu}^2 - \langle T_{k\mu}^2\rangle), \tag{4}$$

where the $\langle\,\rangle$ sign denotes the average with respect to $d\bar{T}$.

Putting expression (4) in expression (3), expanding the fluctuating part of the exponential function and carrying out the integrations over $d\bar{T}$ we obtain

$$P(\{\epsilon_\mu\}) = K\left[\exp\left(-\frac{N+1}{4J^2}\sum \epsilon_\mu^2\right)\right] \prod_{\mu<\nu} |\epsilon_\mu - \epsilon_\nu| \left[1 - \frac{N-1}{N+2}\left(\frac{1}{2J^2} - \frac{M_0^2}{4J^4}\right)\right.$$
$$\left. \times \sum_k \epsilon_k^2 - \frac{N}{N+2}\left(\frac{1}{4J^2} + \frac{M_0^2}{4NJ^4}\right) \sum_{k\neq k'} \epsilon_k\epsilon_{k'} + \ldots\right]. \tag{5}$$

The first term in the square brackets in expression (5) gives rise to Wigner's distribution centred at M_0/N and having the square of dispersion $\frac{1}{4}D^2 = J^2$ for the single eigenvalue (Mehta 1967). By calculating the exact moments of the single eigenvalue E and comparing them with this Wigner distribution, we find that the rest of the terms in expression (5) are just the correction terms of the order of N^{-1}. We can therefore write the distribution of the single eigenvalue as

$$P(E) = K[D^2 - (E - M_0/N)^2]^{1/2}[1 + \alpha(E - M_0/N) + \beta(E - M_0/N)^2$$
$$+ \gamma(E - M_0/N)^3 + \delta(E - M_0/N)^4 + \ldots], \tag{6}$$

where α, β, γ, δ are all terms of the order of N^{-1}. If we keep terms up to $(E - M_0/N)^4$, then they are given by

$$\alpha = -(2/NJ^4)(M_0^3 + 3M_0J^2), \tag{7a}$$
$$\beta = (1/NJ^6)(6J^4 - 2M_0^2J^2 - 3M_0^4), \tag{7b}$$
$$\gamma = (1/NJ^6)(M_0^3 + 3M_0J^2), \tag{7c}$$
$$\delta = -(1/NJ^8)(2J^4 - M_0^2J^2 - M_0^4), \tag{7d}$$
$$K = (2\pi J^2)^{-1}[1 - (1/NJ^4)(2J - M_0^4)], \tag{7e}$$

and reproduce correct moments up to the fourth if one keeps terms of the order of N^{-1} and ignores the higher ones.

We would also like to remark here that the correctness of our procedure can also be checked by calculating the distribution of a component of an N-dimensional unit vector. In this case one can show that the dominant part of the distribution is Gaussian and the complete distribution can be written as a Gaussian plus terms of the order of N^{-1} and higher.

We have, therefore, conclusively established that if the matrix elements are distributed according to expression (1), then the distribution of the single eigenvalue is a Wigner distribution centred at M_0/N plus correction terms of the order of N^{-1}.

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